Uniformization and Characterization of klt varieties

... everything $/\mathbb{C}$

Topic Characterization of singular varieties

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Why Minimal Model Theory: Varieties with K_X purely positive / trivial / purely negative are building blocks.

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Catch Building blocks are singular, with terminal / klt singularities } need to understand those.

§1 : Varieties w/ Kodaira dimension zero

Setting: X projective manifold, $\kappa(X) = 0$

$\rightsquigarrow$ Bochner principle: tensors are controlled by representation theory of holonomy group

Thm (many people) X_{min} has a quasi-étale cover that decomposes

$$X_{min} \longleftarrow (\text{compact torus}) \times \underbrace{(\text{singular } CY)}_{\text{holonomy } SU} \times \underbrace{(\text{singular } HK)}_{\text{holonomy } Sp}$$

Fact Singular HK:
- fundamental group is finite
- étale fundamental group of smooth locus is finite

Conjecture Same is true for singular CY.

Lesson learned singular varieties ? $\rightsquigarrow$ study quasi-étale uniformization

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Thm (EGZ) Every ample class of X_{min} contains a singular Kähler-Einstein metric. □

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§ 2 Varieties of general type

§ 2.1 Recap: the smooth setting

Thm ① (You et al) X a proj. manifold, dim $= n$, K_X ample

$$\Rightarrow \left[2 \cdot (n+1)\, c_2(X) - n \cdot c_1(X)^2 \right] \cdot K_X^{n-2} \geq 0$$

In case of equality, X is a ball quotient. □

M a proj. manifold homeomorphic to X

$\Rightarrow$ M is biholomorphic or conjugate biholomorphic to X. □

Similar results for characterization of tori / hyperelliptic varieties / proj. space

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Thm 2 (Rigidity) X a ball quotient, $\dim X \geq 2$,

M a proj. manifold homeomorphic to X

$\Rightarrow$ M is biholomorphic or conjugate biholomorphic to X. □

Similar results for characterization of tori / hyperelliptic varieties / proj. space

§ 2.1 Recap: smooth setting

Theorem ①: X a projective manifold, $n := \dim X$, K_X ample.

Then: $\left[2 \cdot (n+1)\, c_2(X) - n \cdot c_1(X)^2 \right] \cdot K_X^{n-2} \geq 0$ $\quad (*)$

In case of equality, X is a ball quotient.

Proof (Simpson) Non-Abelian Hodge Theory !!

① $\Omega^1_X \oplus \mathcal{O}_X = \mathcal{E}_X$ is a Higgs bundle!

② Enoki: Ω^1_X is semistable as a vector bundle

$\rightsquigarrow$ $\mathcal{E}_X$ is stable as a Higgs bundle

$\rightsquigarrow$ Bogomolov-Gieseker inequality for $\mathcal{E}_X$ is $(*)$

polystable, $\downarrow$ has van. Chern classes

③ Equality: vanishing of Chern classes for $\mathrm{End}(\mathcal{E}_X) =: \mathcal{F}_X$

$\overset{\text{NAHT}}{\Rightarrow}$ $\mathcal{F}_X$ is flat, connect. comes from VHS

w/ period domain the ball. $\square$

§ 2.1 Recap: smooth setting

Theorem (2): X a ball quotient, $\dim X \geq 2$, M a proj. manifold homeomorphic to X. Then, M is biholomorphic or conjugate biholomorphic to X.

Proof (Siu) Choose homeomorph. $\varphi : M \to X$. Then, homotopy class of φ contains a homeomorph. that is harmonic.

$\rightsquigarrow$ either biholomorphic or conjugate biholomorphic $\square$

§ 2.2 Singular varieties of general type

Goal now: Do all that for singular spaces!

Theorem 1: X n-dimensional, projective, klt and K_X ample. Then

$$\left[2 \cdot (n+1)\, c_2(X) - n \cdot c_1(X)^2 \right] \cdot K_X^{n-2} \geq 0 \qquad \text{„}\mathbb{Q}\text{-Chern Classes"}$$

In case of equality: X is a singular ball quotient.

In particular, X has only quot. sings

Theorem 2: X a singular ball quotient, $\dim X \geq 2$. M normal, projective, homeomorphic to X. Then, M is biholomorphic to X or conjugate biholomorphic to X. Either way, M is a sing. ball quotient.

... similar results for torus quotients, quotients of $\mathbb{P}^n$, quotients of other bounded symmetric domains

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Singular ball quotient: TFAE

- $\exists$ quasi-étale cover of X by a smooth ball quotient
- — " — Galois cover ———— " ————

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§ 2.2 Singular varieties of general type

Theorem 1: X n-dimensional, projective, klt, and K_X ample. Then,

$$\left[2\cdot(n+1)\, c_2(X) - n\cdot c_1(X)^2 \right] \cdot K_X^{n-2} \geq 0$$

In case of equality, X is a singular ball quotient.

What goes into the proof?

① Extension Theorems for diff. forms: if $\tilde{X} \xrightarrow{\pi} X$ is a resolution of sings then, a diff form on $X_{reg} = \tilde{X} \setminus \pi\text{-except. set}$ gives an honest diff. on all of $\tilde{X}$.

Consequences: If $(\Omega^1_X)^{**}$ is loc. free, then X is smooth.

② klt sing. have finite local fundamental groups. $\exists$ a quasi-étale cover $\hat{X} \to X$ where $\hat{\pi}_1(\hat{X})$ and $\hat{\pi}_1(\hat{X}_{reg})$ agree.

$\Rightarrow$ flat bundles on $\hat{X}_{reg}$ extend to flat bundles on $\hat{X}$.

③ Nonabelian Hodge theory for klt varieties

$\rightsquigarrow$ $\Omega^{[1]}_{\hat{X}} \oplus \mathcal{O}_{\hat{X}}$ is a Higgs sheaf.

$\rightsquigarrow$ similar argument as before:

$$\operatorname{End}(\Omega^{[1]}_{\hat{X}} \oplus \mathcal{O}_{\hat{X}}) \text{ is flat on } \hat{X}_{reg}.$$

so: extends to a flat **bundle** on $\hat{X}$

but $\Omega^{[1]}_{\hat{X}}$ is a summand, so locally free, so $\hat{X}$ is smooth, classic result applies. $\square$

§ 2.2 Singular varieties of general type

Theorem 25 X a singular ball quotient, $\dim X \geq 2$.

M projective, normal, homeomorphic to X. Then,

M is biholomorphic or conjugate biholomorphic to X.

Proof Choose an homeomorphism $\varphi : M \to X$. Choose a quasi-étale cover $\gamma : \hat{X} \to X$ where $\hat{X}$ is a unit ball.

$$
\begin{tikzcd}
\tilde{M} \arrow[d, "\text{resolve sings}"'] \arrow[dr, bend left, "\psi"] & \\
\hat{M} \arrow[r] \arrow[d, "s"'] & \hat{X} \arrow[d, "\gamma"] \\
M \arrow[r, "\varphi\text{, homeom.}"'] & X
\end{tikzcd}
$$

theory of Siu applies to this map.

NOW IN BUSINESS.

$\hat{M}$: top. fibre product.

$\hat{X}$: ball quotient.

γ ... branched exactly over X_{sing}.

s: branched over the top. sing. of M

is an analytic set because of Whitney strat. Łojasiewicz.

$\square$

Thank you for your time!